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# Chiral Rings, Superpotentials and the Vacuum Structure of $\mathcal{N} = 1$ Supersymmetric Gauge Theories

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## Abstract

We use the Konishi anomaly equations to construct the exact effective superpotential of the glueball superfields in various  $\mathcal{N} = 1$  supersymmetric gauge theories. We use the superpotentials to study in detail the structure of the spaces of vacua of these theories. We consider chiral and non-chiral  $SU(N)$  models, the exceptional gauge group  $G_2$  and models that break supersymmetry dynamically.

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## 1 Introduction

The strong coupling dynamics of  $\mathcal{N} = 1$  supersymmetric gauge theories in four dimensions is clearly of much theoretical and maybe physical importance. Recently Dijkgraaf and Vafa made a beautiful conjecture [1, 2, 3] that the F-terms of a large class of  $\mathcal{N} = 1$  supersymmetric gauge theories can be computed exactly by a large  $N$  computation in a bosonic matrix model. The assumption is that the relevant fields in the IR are the glueball superfields  $S_i$  and the conjecture provides means of computing their exact effective superpotential. This is done by evaluating the planar diagrams of the matrix model. The generic glueball superpotential is a sum

of Veneziano-Yankielowicz logarithmic superpotential terms [4] and an infinite perturbative sum in the  $S_i$ . Thus, even if the matrix model is not solvable, one can still compute the superpotential to arbitrary power of  $S_i$  by evaluating matrix model diagrams.

In [5] it has been shown for a theory with adjoint matter that the loop equations for the matrix model associated with the  $\mathcal{N} = 1$  gauge theory are equivalent to the generalized Konishi anomaly equations. Besides being a nice observation by itself, the loop equations can sometimes be powerful enough in order to solve the large  $N$  matrix model. Also, one can forget about the matrix model and study the Konishi anomaly equations by themselves. This will be the approach that we will take in this paper.

The aim of this paper is to study various types of  $\mathcal{N} = 1$  supersymmetric gauge theories and to compute the exact glueball effective superpotential by using the Konishi anomaly. We analyze the vacuum structure of those theories. This approach works for chiral and non-chiral theories as well as theories with exceptional gauge groups. One can even study theories with dynamical supersymmetry breaking.

The paper is organized as follows: In section 2 we discuss the general strategy and its limitations. Limitations can be of different sorts. One could be that there are not enough equations to solve for the superpotential. Another requirement is the existence of a supersymmetric vacuum, which limits the analysis of models that break supersymmetry. In section 3 we analyze  $SU(N)$  gauge theory with matter in the fundamental representation and a quartic superpotential. We analyze the different classical and quantum vacua and compute the exact quantum superpotentials. We show how motions in the parameter space of the theory interpolate between different vacua. In section 4 we analyze a gauge theory based on the exceptional group  $G_2$  with matter in the fundamental representation. Again, we compute the exact superpotential and discuss the vacuum structure. In section 5 we discuss a chiral model and perform a similar analysis. Results are in agreement with other methods. In section 6 we analyze the IYIT model that breaks supersymmetry dynamically. In the appendix we prove (under certain conditions) the one-loop exactness of the (generalized) Konishi anomaly equation. This result is used in the previous sections.

## 2 The Classical and Quantum Chiral Rings

In this section we discuss some aspects of the classical and quantum chiral rings of four-dimensional supersymmetric gauge theories. The discussion parallels the one in [5].

We denote the four-dimensional Weyl spinor supersymmetry generators by  $Q_\alpha$  and  $\bar{Q}_{\dot{\alpha}}$ . Chiral operators are operators annihilated by  $\bar{Q}_{\dot{\alpha}}$ . For instance, the lowest component  $\phi$  of a chiral superfield  $\Phi$  is a chiral operator. The OPE of two chiral operators is nonsingular and allows for the definition

of the product of two chiral operators. The product of chiral operators is also a chiral operator. Furthermore, one can define a ring structure on the set of equivalence classes of chiral operators modulo operators of the form  $\{\bar{Q}_{\dot{\alpha}}, \dots\}$ .

Consider a general  $\mathcal{N} = 1$  supersymmetric gauge theory with gauge group  $G$  and some matter supermultiplets. Denote by  $V$  the vector superfield in the adjoint representation of  $G$ , by  $\Phi$  chiral superfields in a representation  $r$  of  $G$  and by  $\phi$  their lowest component. The field strength (spinor) superfield is  $W_{\alpha} = -\frac{1}{4}\bar{D}^2 e^{-V} D_{\alpha} e^V$  and is a chiral superfield. Using products of  $\phi$  and  $W_{\alpha}$  we construct chiral operators<sup>1</sup>. They satisfy the relation

$$W_{\alpha}^{(r)} \phi^{(r)} = 0 \quad (2.1)$$

modulo  $\{\bar{Q}_{\dot{\alpha}}, \dots\}$  terms, where we noted that  $\phi$  transforms in a representation  $r$  of the gauge group  $G$ .  $W_{\alpha}^{(r)} = W_{\alpha}^a T^a(r)$  with  $T^a(r)$  being the generators of the gauge group  $G$  in the representation  $r$ . The relation (2.1) implies in particular

$$\{W_{\alpha}^{(r)}, W_{\beta}^{(r)}\} = 0 \quad (2.2)$$

We will be interested in the sector of gauge invariant chiral operators. These can be constructed as gauge invariant composites of  $W_{\alpha}$  and  $\phi$  taking the identity (2.1) into account. We will call the chiral ring the ring of equivalence classes of gauge invariant chiral operators modulo  $\{\bar{Q}_{\dot{\alpha}}, \dots\}$ . An important element of the chiral ring is the glueball superfield  $S$

$$S = -\frac{1}{32\pi^2} \text{Tr} W^2 \quad (2.3)$$

The gauge invariant chiral operators made of the matter multiplets parametrize the moduli space of vacua of the supersymmetric gauge theory. It is therefore of interest to find the relations among the elements of the chiral ring which constrain the structure of the moduli space. These relations can be different in the classical and in the quantum theory.

## 2.1 The Classical Chiral Ring Relations

Let us comment first on the relations in the classical chiral ring. There are two types of relations. The first are kinematic ones which are associated with group theory and statistics and contain no dynamics of the classical theory. One such relation is

$$S^{\dim G + 1} = 0 \quad (2.4)$$

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<sup>1</sup>In this paper we denote by  $W_{\alpha}$  the supersymmetric field strength as well as its lowest component, the gaugino.

or an even stronger relation

$$S^{h^\vee} = 0 , \quad (2.5)$$

where  $h^\vee$  is the dual Coxeter number of the group  $G$ . The last relation has been proven in [5] for  $G = SU(N)$  and in [6] for gauge groups  $Sp(N)$  and  $SO(N)$ . It has been conjectured to hold for all simple groups. Another example of a kinematic relation is

$$\text{Tr} \phi^n = \mathcal{P}(\text{Tr} \phi, \dots, \text{Tr} \phi^N) , \quad (2.6)$$

with  $\phi$  an adjoint field in a  $U(N)$  gauge theory and  $n > N$ .

The second type of relations in the classical ring are the dynamical relations given by the variation of the tree level superpotential  $W_{tree}$

$$\frac{\partial W_{tree}}{\partial \phi} = 0 . \quad (2.7)$$

These relations are not gauge invariant but can be implemented in a gauge invariant way. For instance by

$$\phi \frac{\partial W_{tree}}{\partial \phi} = 0 , \quad (2.8)$$

with appropriate extraction of the gauge invariant parts of the equations.

For a generic tree level superpotential the relations (2.7) fix the moduli space of the theory up to a discrete choice. This means that we can solve these relations (possibly together with kinematic relations) and fix all the gauge invariant chiral operators made out of matter fields.

## 2.2 The Quantum Chiral Ring Relations

The classical chiral ring relations have quantum deformations. In general it is hard to find the quantum deformations unless there are enough symmetries in the theory. However the classical relations arising from (2.7) have a natural generalization as anomalous Ward identities of the quantized matter sector in a classical gauge(ino) background. If  $\phi$  transforms in a representation  $r$  of the gauge group  $G$ , then the classical superpotential relation (2.7) transforms in the dual representation  $\bar{r}$ . It has to be contracted with a chiral operator  $\phi'$  in a representation  $r'$  such that the decomposition of the tensor product  $\bar{r} \otimes r'$  contains a singlet representation. This yields a classical chiral ring relation

$$\phi' \frac{\partial W_{tree}}{\partial \phi} = 0 . \quad (2.9)$$

This relation can be interpreted as a classical Ward identity for the Konishi current  $J = \Phi^\dagger e^V \Phi'$

$$\bar{D}^2 J = \phi' \frac{\partial W_{tree}}{\partial \phi} . \quad (2.10)$$

This Ward identity gets an anomalous contribution in the quantum theory. In general  $\phi'$  is a function of  $\phi$  and the generalized Konishi anomaly takes the form [5, 7, 8]

$$\bar{D}^2 J = \phi'_i \frac{\partial W_{\text{tree}}}{\partial \phi_i} + \frac{1}{32\pi^2} W_{\alpha i}{}^j W^{\alpha}{}_j{}^k \frac{\partial \phi'_k}{\partial \phi_i}, \quad (2.11)$$

where  $i, j$  and  $k$  are gauge indices and their contraction is in the appropriate representation. This Ward identity has tree level and one loop contributions. In order to prove that there are no higher loop or nonperturbative corrections to this identity one has to use symmetry arguments and asymptotic behavior in the coupling constants (see the appendix for a proof under certain conditions).

Since the divergence  $\bar{D}^2 J$  is  $\bar{Q}$ -exact it vanishes in a supersymmetric vacuum. Taking the Wilsonian expectation value of (2.11) in a slowly varying gaugino background, we get

$$\left\langle \phi'_i \frac{\partial W_{\text{tree}}}{\partial \phi_i} \right\rangle_S + \left\langle \frac{1}{32\pi^2} W_{\alpha i}{}^j W^{\alpha}{}_j{}^k \frac{\partial \phi'_k}{\partial \phi_i} \right\rangle_S = 0. \quad (2.12)$$

This relation will be our main tool to determine the effective superpotential.

### 2.3 The Effective Superpotential

We will be interested in determining the effective superpotential  $W_{\text{eff}}$  for the glueball superfield  $S$  with the matter fields  $\Phi$  being integrated out.

The strategy we will use is as follows. We first use the gradient equations for  $W_{\text{eff}}$  in the tree level superpotential couplings. For a tree level superpotential

$$W_{\text{tree}} = \sum_I g_I \sigma_I, \quad (2.13)$$

where  $\sigma_I$  are gauge invariant chiral operators, we get

$$\frac{\partial W_{\text{eff}}}{\partial g_I} = \langle \sigma_I \rangle_S. \quad (2.14)$$

The expectation values are taken in a slowly varying (classical) gaugino background. We then use the chiral ring relations (2.12) to solve for the  $\langle \sigma_I \rangle_S$  in terms of the  $S$  and the coupling constants  $g_I$ . In order to solve these relations we use the factorization property

$$\langle \sigma_I \sigma_J \rangle_S = \langle \sigma_I \rangle_S \langle \sigma_J \rangle_S \quad (2.15)$$

of expectation values of chiral operators in a supersymmetric vacuum.

We insert the solutions into the gradient equations (2.14) and determine the effective superpotential up to a function  $C(S)$ , which does not depend

on the  $g_I$ . We can determine this function by semi classical arguments in certain limits of the coupling constant space, where the low energy dynamics is described by pure SYM. The strong IR gauge dynamics is then captured by a Veneziano-Yankielowicz type superpotential. If there are several such limit points, there are consistency checks one can do.

### 3 $SU(N_c)$ with Fundamental Matter

In this section we will consider  $\mathcal{N} = 1$  supersymmetric gauge theories with  $SU(N_c)$  gauge group and matter in the fundamental representation. We will use SQCD with one flavor as a representative model to outline our technique. Due to the small number of generators in the chiral ring the usual Konishi anomaly suffices to obtain the effective superpotential. Using the full effective superpotential we will then analyze the vacuum structure of the model. We discuss also various generalizations.

#### 3.1 $SU(N_c)$ with One Flavor

Let us start by considering SQCD with gauge group  $SU(N_c)$  and one flavor. This theory was studied in the matrix model context in [9]-[28]. The theory is non-chiral and has one fundamental matter multiplet  $Q$  and one antifundamental matter multiplet  $\tilde{Q}$ . There are only two gauge invariant chiral operators one can build out of the fundamental fields, the meson  $M = \tilde{Q}Q$  and the gaugino bilinear  $S$ . To begin with let us assume the tree level superpotential

$$W_{tree} = mM + \lambda M^2 . \quad (3.1)$$

Note we have chosen a rather simple superpotential to illustrate our method but in principle we could take an arbitrary polynomial in the meson field. The theory with (3.1) has two classical vacua at  $M = 0$  and  $M = -\frac{m}{2\lambda}$ . In the first vacuum the gauge group is unbroken, whereas in the second vacuum the gauge group is broken to  $SU(N_c - 1)$ .

The Konishi variation  $\delta Q = \epsilon Q$  leads to the relation

$$m\langle M \rangle_S + 2\lambda\langle M^2 \rangle_S = S . \quad (3.2)$$

We also have the relations

$$\frac{\partial W_{eff}}{\partial m} = \langle M \rangle_S , \quad \frac{\partial W_{eff}}{\partial \lambda} = \langle M^2 \rangle_S . \quad (3.3)$$

If we use the factorization properties for the chiral operators we get a quadratic equation for  $\langle M \rangle_S$

$$2\lambda\langle M \rangle_S^2 + m\langle M \rangle_S - S = 0 , \quad (3.4)$$

Now we get two differential equations which control the dependence of the effective superpotential on the bare couplings

$$\frac{\partial W_{\text{eff}}}{\partial m} = -\frac{m}{4\lambda} \pm \sqrt{\frac{m^2}{16\lambda^2} + \frac{S}{2\lambda}} , \quad (3.5)$$

$$\frac{\partial W_{\text{eff}}}{\partial \lambda} = \left( -\frac{m}{4\lambda} \pm \sqrt{\frac{m^2}{16\lambda^2} + \frac{S}{2\lambda}} \right)^2 . \quad (3.6)$$

By taking the classical limit  $S \rightarrow 0$  in the above equations we find that the  $+$  sign corresponds to the classical vacuum  $M = 0$  and the  $-$  sign to  $M = -\frac{m}{2\lambda}$ . These two equations can be integrated to give the following superpotential

$$W_{\text{eff}} = -\frac{m^2}{8\lambda} \pm \frac{m^2}{8\lambda} \sqrt{1 + \frac{8\lambda}{m^2} S} + S \log m + S \log (1 \pm \sqrt{1 + \frac{8\lambda}{m^2} S}) + C(S) , \quad (3.7)$$

where  $C(S)$  is an  $S$  dependent integration constant.

To determine  $C(S)$  we proceed as follows. In the classical vacuum  $M = 0$  the matter fields have mass  $m$ . If we take  $m \gg \Lambda$ , where  $\Lambda$  is the dynamically generated mass scale we can integrate out the matter fields in perturbation theory. Hence, we separate the pure gauge dynamics from the dynamics of the matter fields. We will take care of the pure gauge dynamics in the strong coupling regime by an appropriate Veneziano-Yankielowicz term [4]. We concentrate first on an effective action  $W_{\text{eff}}^{\text{pert.}}(S)$  for  $S$  obtained by integrating out the matter fields in perturbation theory. As explained in [5] the terms of  $W_{\text{eff}}^{\text{pert.}}(S)$  linear in  $S$  come from integrating out the matter fields at one loop. Higher loops depend on the bare couplings in the tree level superpotential and are thus already included in (3.7) [29]. The contribution  $C(S)$  can thus be determined by an explicit one loop calculation [5].

Note that in general this requires that we have a classical vacuum where all the matter fields are massive around which we can do perturbation theory. However, this method also works for vacua where all the matter degrees of freedom are eaten up by the Higgs mechanism. This will prove especially useful in the case of chiral models where we cannot have mass terms for the matter fields.

Right now we consider perturbation theory around the classical vacuum  $M = 0$ . The perturbative superpotential at an energy scale  $\Lambda < \mu < m$  is given by

$$W_{\text{eff}}^{\text{pert.}} = \tau_0 S + 3N_c S \log \frac{\Lambda_0}{\mu} + S \log \frac{m}{\Lambda_0} + O(S^2) , \quad (3.8)$$

where  $\Lambda_0$  is the UV cutoff. Substituting

$$\tau_0 = -(3N_c - 1) \log \frac{\Lambda_0}{\Lambda} \quad (3.9)$$



amounts to replacing  $\Lambda_0$  by  $\Lambda$  in (3.8). Since we compare (3.7) to a one loop calculation around the vacuum  $M = 0$  we have to choose the branch with the + sign. Matching then the contributions of  $O(S)$  in (3.7) and (3.8) gives

$$C(S) = 3N_c S \log \frac{\Lambda}{\mu} - S \log \Lambda - \frac{S}{2} - S \log 2 . \quad (3.10)$$

We have to include the strong coupling dynamics by replacing <sup>2</sup>

$$3N_c S \log \frac{\Lambda}{\mu} \rightarrow N_c S \left( -\log \frac{S}{\Lambda^3} + 1 \right) . \quad (3.11)$$

Finally matching the scale  $\hat{\Lambda}$  of the pure SYM according to

$$\hat{\Lambda}^{3N_c} = \Lambda^{3N_c-1} m , \quad (3.12)$$

we find the full nonperturbative superpotential

$$\begin{aligned} W_{eff} = & N_c S \left( -\log \frac{S}{\hat{\Lambda}^3} + 1 \right) - \frac{S}{2} - \frac{m^2}{8\lambda} \pm \frac{m^2}{8\lambda} \sqrt{1 + \frac{8\lambda}{m^2} S} + \\ & + S \log \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right) . \end{aligned} \quad (3.13)$$

#### *Alternative Derivation of $W_{eff}$*

To gain confidence in this result, we can give an alternative derivation of (3.13) based upon the ILS linearity principle [30]. If we consider the tree level superpotential as a perturbation to the low energy physics we can first forget about the superpotential and consider  $SU(N_c)$  SYM with one massless flavor. Along the flat direction parametrized by  $M$  the gauge group is generically broken to  $SU(N_c-1)$ . We thus expect an appropriate effective description in terms of a pure  $SU(N_c-1)$  SYM obtained by Higgsing the original  $SU(N_c)$  with one flavor. The effective superpotential for the  $SU(N_c-1)$  theory is just the Veneziano-Yankielowicz superpotential

$$W_{eff} = (N_c - 1) S \left[ -\log \frac{S}{\tilde{\Lambda}^3} + 1 \right] . \quad (3.14)$$

We have to relate the scale  $\tilde{\Lambda}$  of the Higgsed theory to the scale  $\Lambda$  of the original theory. This is done at the scale set by the meson expectation value  $M$ . We have

$$\left( \frac{\tilde{\Lambda}}{M^{1/2}} \right)^{3(N_c-1)} = \left( \frac{\Lambda}{M^{1/2}} \right)^{3N_c-1} , \quad (3.15)$$

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<sup>2</sup>This amounts to replacing  $\mu^3 \rightarrow S/e$ .

such that

$$\tilde{\Lambda}^{3(N_c-1)} = \frac{\Lambda^{3N_c-1}}{M} . \quad (3.16)$$

Adding the tree level potential will localize the meson expectation value at the quantum vacuum. This localization is equivalent to integrating out  $M$  from the effective superpotential. However, the quantum expectation value of the meson as a function of  $S$  is given by the Konishi relation (3.5). So if we add the tree level superpotential (3.1) to (3.14) and replace  $M$  by the quantum expectation value  $\langle M \rangle_S$  given by the Konishi relation as

$$\langle M \rangle_S = -\frac{m}{4\lambda} \pm \sqrt{\frac{m^2}{16\lambda^2} + \frac{S}{2\lambda}} , \quad (3.17)$$

we exactly reproduce the full nonperturbative superpotential given in (3.13). This gives us a nice consistency check.

#### Relation to the Vector Model

The anomaly equation (3.4) can also be derived from the zero dimensional vector model

$$\int dQ d\tilde{Q} e^{-\frac{1}{g_s} W_{tree}(\tilde{Q}Q)} . \quad (3.18)$$

The Ward identity for the variation  $Q \mapsto Q + \epsilon Q$  is

$$g_s N_{VM} = \left\langle \frac{\partial W_{tree}}{\partial Q} Q \right\rangle_{VM} = m \langle M \rangle_{VM} + 2\lambda \langle M^2 \rangle_{VM} . \quad (3.19)$$

Making the identification  $S = g_s N_{VM}$ , one reproduces the anomaly equation. In the planar limit  $\langle M^2 \rangle_{VM}$  factorizes and we get the same result as before.

### 3.2 The Vacuum Structure

The expression for the effective superpotential has two branches for the two signs of the square root. These correspond to the two classical vacua. One can make an expansion for small  $\frac{8\lambda}{m^2} S$  in both branches to recover  $N_c$  vacua in the one branch and  $N_c - 1$  vacua in the other. This is the expected result from the semiclassical analysis since we expect the unbroken gauge symmetries to confine in the IR.

The quantum vacua are at the critical points of  $W_{eff}$ , *i.e.* they satisfy

$$\log \left[ \frac{\hat{\Lambda}^{3N_c}}{S^{N_c}} \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right) \right] = 0 . \quad (3.20)$$

This can be simplified to

$$\hat{S}^{2N_c-1} - \hat{S}^{N_c-1} - z = 0 , \quad \hat{S} = \frac{S}{\hat{\Lambda}^3} , \quad z = \hat{\Lambda}^3 \frac{2\lambda}{m^2} . \quad (3.21)$$

From this equation it is possible to understand the quantum parameter space. It is given as the complex surface, associated to the analytic continuation in  $z$  of the gaugino condensate  $S = \hat{\Lambda}^3 \hat{S}(z)$ .

Naturally, from the polynomial equation we expect  $2N_c - 1$  sheets. There is an order  $2N_c - 1$  branching point at  $z = \infty$  and an order  $N_c - 1$  branching point at  $z = 0$ . The remaining  $N_c$  branching points are double points located at the roots of the equation

$$z^{N_c} = (-)^{N_c} \frac{(N_c - 1)^{N_c - 1} N_c^{N_c}}{(2N_c - 1)^{2N_c - 1}}. \quad (3.22)$$

At the double points two zeros of the polynomial (3.21) coincide, such that its first derivative vanishes. Since this is equivalent to the second derivative of the superpotential, we generally expect massless glueballs at these points. This implies that the mass gap disappears at those points, unless the Kähler potential also gets singular.

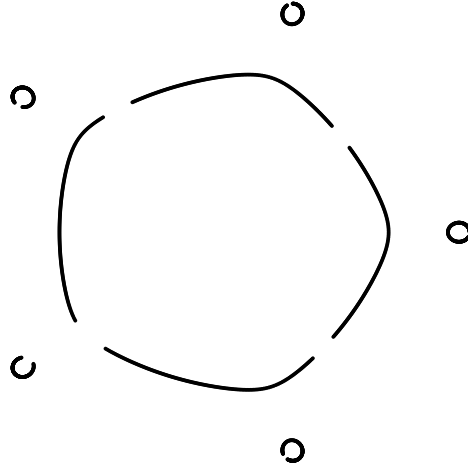


Figure 1: Monodromy around  $z = 0$  for  $G = SU(5)$ : we follow the positions of the vacua in the  $S$  plane as we take a small, not completely closed circle around  $z = 0$ . The un-Higgsed vacua, corresponding to the smaller outer circles, are not interchanged, whereas the big circle corresponds to the four Higgsed vacua which get interchanged as we change the phase of  $z$ .

At the point  $z = 0$  we expect to find a clear distinction of Higgsed and un-Higgsed quantum vacua. And actually for small  $z$  the above equation factorizes to  $S^{N_c - 1} - \tilde{\Lambda}^{3(N_c - 1)} = 0$  (where we used  $z\hat{\Lambda}^{3(N_c - 1)} = \tilde{\Lambda}^{3(N_c - 1)}$ ) and  $S^{N_c} - \hat{\Lambda}^{3N_c} = 0$ . These give  $N_c - 1$  vacua for the Higgsed branch and  $N_c$  for the un-Higgsed one with the appropriate scales  $\tilde{\Lambda}$  and  $\hat{\Lambda}$ .

Circling  $z = 0$  corresponds to rotating the scale  $\tilde{\Lambda}^{3(N_c - 1)}$  and thus changes from one quantum vacuum to the next in the Higgsed branch (see

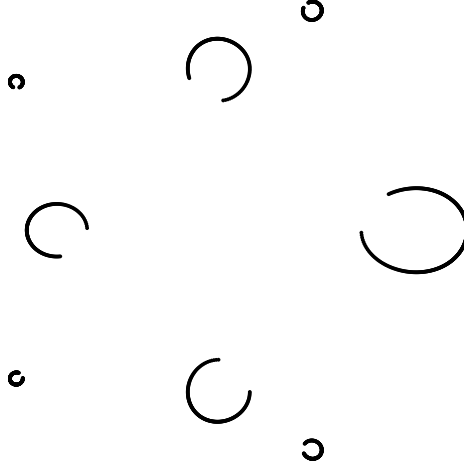


Figure 2: Here we can see the motion of the vacua of the  $SU(5)$  theory in the  $S$  plane as we vary the parameter  $z$  along a small circle around a root of (3.22). The un-Higgsed vacua correspond to the smaller outer circles, whereas the bigger, inner circles correspond to the Higgsed vacua. Most interesting is the rightmost big circle in which we can see the exchange of an un-Higgsed with a Higgsed vacuum.

Fig. 1). Circling the bulk branch points changes from quantum vacua in the Higgsed branch to the un-Higgsed one (see Fig. 2). If we take  $z$  to be large, then the  $2N_c - 1$  vacua arrange themselves symmetrically on a circle and the monodromy at infinity  $z \rightarrow ze^{2\pi i}$  exchanges them in a  $Z_{2N_c-1}$  symmetric manner. So at  $z = \infty$  both branches look similar, which seems natural, as the minima of  $W_{tree}$  degenerate ( $z \rightarrow \infty$  is like  $m \rightarrow 0$ ). The structure of the quantum parameter space for other models has been discussed recently in [31, 32, 33].

#### The Massless Limit

It is interesting to analyze the massless limit of our SQCD model. As stated above this corresponds to the limit  $z \rightarrow \infty$  in the quantum parameter space. From the analysis of the quantum parameter space we expect that the two branches join to give the  $2N_c - 1$  vacua. Indeed, the effective superpotential (3.13) has a finite  $m \rightarrow 0$  limit on both branches and we can recover the  $2N_c - 1$  vacua from either branch. If we start on the un-Higgsed branch (the  $+$  branch of (3.13)) and take the massless limit we obtain

$$W_{eff} = S \log \Lambda^{3N_c-1} + \left(N_c - \frac{1}{2}\right) S - \left(N_c - \frac{1}{2}\right) S \log S + \frac{1}{2} S \log \lambda + \frac{1}{2} S \log 2. \quad (3.23)$$

Minimizing with respect to  $S$  gives the  $2N_c - 1$  vacua

$$S = e^{\frac{4\pi i k}{2N_c-1}} (2\lambda \Lambda^{6N_c-2})^{\frac{1}{2N_c-1}}. \quad (3.24)$$

We can compare this result with the Affleck-Dine-Seiberg analysis of the same system. In this approach the exact effective superpotential<sup>3</sup> in the massless case is given by

$$W_{eff}^{ADS} = (N_c - 1) \left( \frac{\Lambda^{3N_c-1}}{M} \right)^{\frac{1}{N_c-1}} + \lambda M^2 . \quad (3.25)$$

Looking for the mesonic vacua we find

$$M = \left( \frac{\Lambda^{3N_c-1}}{2\lambda} \right)^{\frac{1}{2N_c-1}} e^{\frac{2\pi i k}{2N_c-1}} . \quad (3.26)$$

As expected the vevs (3.24) and (3.26) satisfy the Konishi relation

$$2\lambda \langle M \rangle_S^2 - S = 0 , \quad (3.27)$$

obtained from (3.4) in the massless limit. We thus get a nice picture of the  $2N_c - 1$  vacua of the massless model if we think of them as obtained in the massless limit of a massive model.

### 3.3 $SU(N_c)$ with $N_c \geq N_f > 1$

We will now turn to a generalization of the above procedure to the case of  $N_c \geq N_f > 1$ , with a simple tree level superpotential. The effective superpotential can be derived again with the use of the Konishi relations. With the superpotential at hand, we investigate the vacuum structure of the model. Many of the classical vacua turn out to be connected in the quantum parameter space.

The model we consider is  $SU(N)$  SQCD with  $N_f$  flavors and tree level superpotential,

$$W_{tree} = m \operatorname{tr} M + \lambda \operatorname{tr} M^2 , \quad (3.28)$$

with  $M$  the meson  $M_I^J = \tilde{Q}_I^i Q_i^J$ . This superpotential breaks the  $U(N_f) \times U(N_f)$  flavor symmetry to a diagonal  $SU(N_f)$ . The diagonally embedded  $U(1)_B$  is responsible for the baryon number conservation.

The classical vacua can be understood in terms of the meson field  $M$ . First, we rotate the meson matrix  $M_I^J$  to diagonal form by flavor rotations. Then the tree level superpotential allows that  $N_f^+$  eigenvalues sit at zero and  $N_f^-$  at the minimum  $M_I^I = -m/2\lambda$ . (With the condition  $N_f^+ + N_f^- = N_f$ .) In total we have  $N_f$  choices to distribute the eigenvalues of the meson.

---

<sup>3</sup>The ADS superpotential can also be obtained from our glueball superpotential by a Legendre transform in  $m$  and subsequent integrating out of  $S$ .

Starting from a classical vacuum  $(N_f^+, N_f^-)$  we have a clear expectation of the quantum theory for the parameter in the range  $m/\lambda \gg \Lambda^2$  and  $m \gg \Lambda$ . That is, for energies much higher than the meson expectation values  $M_I^I = -m/2\lambda$  and the squark masses  $m$ , we expect to see  $SU(N_c)$  gauge dynamics with  $N_f$  almost massless quarks. (We assume an appropriate UV-completion of the above tree level potential.) Lowering the scale below the squark masses and the meson expectation values, but still above  $\Lambda$ , we expect to find pure  $SU(N_c - N_f^-)$  supersymmetric gauge dynamics with scale  $\tilde{\Lambda}^{3(N_c - N_f^-)} = \Lambda^{3N_c - N_f} m^{N_f^+} (2\lambda/m)^{N_f^-}$ . Finally, for energies below  $\tilde{\Lambda}$  one finds confinement with  $N_c - N_f^-$  supersymmetric vacua. Starting from this well known vacuum structure we will extend the knowledge of the vacuum structure to the case  $m/\lambda < \Lambda^2$  and  $m < \Lambda$  in the following.

To this end we have to use the Konishi relations. The flavor dependent Konishi anomaly variation  $\delta Q_i^I = \lambda^I{}_J Q_i^J$  leads to

$$m M_I^J + 2\lambda(M^2)_I^J = \delta_I^J S. \quad (3.29)$$

We can solve for the diagonal entries of the meson  $M$ . Here we pick  $N_f^+$  eigenvalues to converge to the vacuum  $M = 0$  and  $N_f^-$  eigenvalues to converge to  $M = -m/2\lambda$  in the classical limit,  $S \rightarrow 0$ . This amounts to choosing branches for each of the eigenvalues in (3.29). The traces then have the vacuum expectation values

$$\begin{aligned} \langle \text{tr} M \rangle_S &= -N_f^+ \frac{m}{2\lambda} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right) - N_f^- \frac{m}{2\lambda} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right) \\ \langle \text{tr} M^2 \rangle_S &= N_f^+ \frac{m^2}{4\lambda^2} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right)^2 + N_f^- \frac{m^2}{4\lambda^2} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right)^2. \end{aligned}$$

Note that this will break the diagonal  $SU(N_f)$  flavor symmetry to  $SU(N_f^+) \times SU(N_f^-) \times U(1)$ , leaving  $2N_f^+ N_f^-$  massless Goldstone bosons.

In order to find the effective superpotential we can integrate the two gradient equations

$$\frac{\partial W_{eff}}{\partial m} = \langle \text{tr} M \rangle_S, \quad \frac{\partial W_{eff}}{\partial \lambda} = \langle \text{tr} M^2 \rangle_S. \quad (3.30)$$

By matching the integration constant, such that the appropriate VY potential is reproduced in the limit  $m/\lambda \gg \Lambda^2$  and  $m \gg \Lambda$ , we can determine the effective superpotential

$$\begin{aligned} W_{eff} &= N_c S \left( -\log \frac{S}{\tilde{\Lambda}^3} + 1 \right) - N_f \left( \frac{S}{2} + \frac{m^2}{8\lambda} \right) + (N_f^+ - N_f^-) \frac{m^2}{8\lambda} \sqrt{1 + \frac{8\lambda}{m^2} S} + \\ &+ S \log \left[ \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right)^{N_f^-} \right]. \end{aligned} \quad (3.31)$$

This effective superpotential does not depend on the massless Goldstone modes, since they are true moduli by symmetry. This is like integrating out the radial direction in a Mexican hat potential.

The scales in this model are the UV scale  $\Lambda$ , the scale  $\hat{\Lambda}$  for the theory with  $N_f$  massive quarks around  $Q = 0$ , and the scale  $\tilde{\Lambda}$  for the theory with  $N_f^+$  massive quarks around  $Q = 0$  and  $N_f^-$  Higgsing quarks,

$$\hat{\Lambda}^{3N_c} = \Lambda^{3N_c - N_f} m^{N_f} = \tilde{\Lambda}^{3(N_c - N_f^-)} (m^2/2\lambda)^{N_f^-} . \quad (3.32)$$

### *The Vacuum Structure*

Let us understand this result better in the limit of small  $S$ . One finds  $N_c - N_f^-$  vacua with the appropriate scale plugged in. By analytic continuation in the parameter space, we can change the sign of the square roots, *i.e.* exchange the role of  $N_f^+$  and  $N_f^-$ . Then for small  $S$  we find  $N_c - N_f^+$  vacua corresponding to  $N_f^+$  Higgsing squarks. As in the case  $N_f = 1$ , these classical vacua are smoothly connected in the quantum parameter space.

The critical points of  $W_{eff}$  are the supersymmetric ground states of the theory. For a given branch of  $W_{eff}$  they are given by the following equation for  $S$ ,

$$\log \left[ \frac{\hat{\Lambda}^{3N_c}}{S^{N_c}} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda}{m^2} S} \right)^{N_f^-} \right] = 0 . \quad (3.33)$$

For  $N_f < N_c$  this has  $N_c - N_f^-$  solutions. However, we have to take into account that the  $+$  and  $-$  branches can be distributed in  $\binom{N_f}{r}$  different ways on the  $N_f$  meson eigenvalues. Hence, the total number of vacua is

$$\sum_{r=0}^{N_f} (N_c - r) \binom{N_f}{r} = (2N_c - N_f) 2^{N_f-1} . \quad (3.34)$$

### *The Case $N_f = N_c$*

We would like to discuss now the above results for the special case  $N_f = N_c$ . More specifically we consider the two branches:  $N_f^- = 0$  and  $N_f^- = N_c$ . In the first branch all meson vevs are zero at the classical level, the gauge group is unbroken and confines in the IR giving rise to  $N_c$  vacua. In the second case all meson vevs are non-zero, the gauge group is broken to nothing and classically there is a unique ground state.

Now we look for the quantum vacua by analyzing (3.33). It turns out that the  $N_f^- = 0$  case has  $N_c$  solutions and the vevs are given by

$$\begin{aligned} S &= e^{2\pi i k/N_c} \hat{\Lambda}^3 \left( 1 + e^{2\pi i k/N_c} \frac{2\lambda \Lambda^2}{m} \right), \\ \langle \text{tr} M \rangle_S &= e^{2\pi i k/N_c} N_f \Lambda^2, \end{aligned} \quad (3.35)$$

with  $\Lambda^2 m = \hat{\Lambda}^3$ . In the limit of small  $\frac{2\lambda \Lambda^2}{m}$  the gaugino condensate reduces to  $S = e^{2\pi i k/N_c} \hat{\Lambda}^3$ , the vacua of pure  $SU(N_c)$  gauge dynamics. There are  $N_c$  points  $m/2\lambda = -e^{2\pi i k/N_c} \Lambda^2$ , where the gluino condensate vanishes.

The other case with  $N_f^- = N_c$  is more subtle but a careful analysis of (3.33) shows that there is an extremum at  $S = 0$  under the condition that  $\left(-\frac{2\lambda \hat{\Lambda}^3}{m^2}\right)^{N_c} = 1$ . These are no new solutions, but just the points with  $S = 0$  from (3.35). The fact that the gauge group is completely broken on the  $N_f^- = N_c$  branch is consistent with the vanishing of the gluino condensate.

We see that the full parameter space has  $N_c$  sheets, where each of them has two distinguished points, one corresponding to a vacuum of pure  $SU(N_c)$  gauge dynamics and the other to a fully Higgsed vacuum. We will find this structure useful when considering dynamical SUSY breaking.

We would like to close this section with the observation, that the expectation value of the meson  $M$  satisfies<sup>4</sup>

$$\text{Det } M = \Lambda^{2N_c}, \quad (3.36)$$

all over the parameter space. On the other hand we can consider the generalized Konishi variation  $\delta Q_i^I = \epsilon_{ij_2 \dots j_{N_c}} \epsilon^{IJ_2 \dots J_{N_c}} \tilde{Q}_{J_2}^{j_2} \dots \tilde{Q}_{J_{N_c}}^{j_{N_c}}$  which leads to the relation

$$(mN_c + 2\lambda \text{tr} M)B = 0, \quad (3.37)$$

where  $B = \det Q$  is the baryon. This implies  $B = 0$ . Similarly, we can show that  $\tilde{B} = 0$ . This shows the validity of the relation

$$\det M - B\tilde{B} = \Lambda^{2N_c}. \quad (3.38)$$

### 3.4 More General $W_{\text{tree}}$

Let us now illustrate how to implement more general tree level superpotentials

$$W_{\text{tree}}(M) = \sum_{j=1}^n \frac{g_j}{j} M^j. \quad (3.39)$$

---

<sup>4</sup>Although we have discussed only the extremal cases  $N_f^- = 0$  and  $N_f^- = N_c$  the quantum constraint from (3.33) can be verified also for a generic eigenvalue distribution for  $N_f = N_f^+ + N_f^- = N_c$ .



in SQCD with  $N_f = 1$ . In that case the Konishi constraint yields

$$\sum_{j=1}^n g_j \langle M \rangle_S^j = S . \quad (3.40)$$

This equation has  $n$  solutions for  $\langle M \rangle_S$ . Inserting this into the gradient equations

$$\frac{\partial W_{eff}}{\partial g_j} = \frac{1}{j} \langle M \rangle_S^j \quad (3.41)$$

one can solve for  $W_{eff}$ . To see that these gradient equations are integrable, we have to show, that there is no curl in the flow. First note

$$0 = \frac{\partial}{\partial g_k} \left( \sum_j g_j \langle M \rangle_S^j - S \right) = \left( \sum_j j g_j \langle M \rangle_S^{j-1} \right) \frac{\partial \langle M \rangle_S}{\partial g_k} + \langle M \rangle_S^k . \quad (3.42)$$

Using this we get

$$\frac{\partial}{\partial g_k} \frac{1}{j} \langle M \rangle_S^j = - \frac{\langle M \rangle_S^{j+k-1}}{\sum_l l g_l \langle M \rangle_S^{l-1}} . \quad (3.43)$$

This shows that the flow is integrable. The integral is again fixed up to a function only of  $S$ , which has to be fixed by asymptotic behavior. Here we have  $n$  asymptotic regions. One asymptotic region has unbroken  $SU(N_c)$  gauge group, *i.e.*  $N_c$  confining vacua. In each of the other asymptotic regions the gauge group is Higgsed down to  $SU(N_c - 1)$ , *i.e.* there are  $N_c - 1$  vacua, giving rise to a total of  $n(N_c - 1) + 1$  vacua.

### *The Massless Limit Revisited*

We can now use a more general tree level superpotential to calculate the effective superpotential for the massless case. We use a technique that will be crucial in dealing with dynamical supersymmetry breaking and with chiral models where no mass term is possible.

Our aim is to calculate the effective superpotential for a tree level superpotential

$$W_{tree} = \lambda M^2 . \quad (3.44)$$

One possibility in this model is to add a mass term, apply our technique and then send  $m \rightarrow 0$  as we have already done in a previous section. The other possibility which is applicable also for chiral models, is to add a tree level term which gives a classical vacuum where the gauge group is Higgsed. We will take

$$W_{tree} = \lambda M^2 + \alpha M^4 . \quad (3.45)$$

The classical vacua are then  $M^2 = 0$  and  $M^2 = -\lambda/2\alpha$ . If we solve the Konishi relations as usual we get (for the Higgsed branch)

$$W_{eff} = -\frac{\lambda^2}{8\alpha} - \frac{\lambda^2}{8\alpha} \sqrt{1 + \frac{4\alpha}{\lambda^2} S} + 1/2 S \log \alpha - \\ -1/2 S \log \lambda - 1/2 S \log \left(1 + \sqrt{1 + \frac{4\alpha}{\lambda^2} S}\right) + C(S) . \quad (3.46)$$

On the Higgsed branch we can easily match it to the Veneziano-Yankielowicz potential for  $SU(N_c - 1)$ , whereas on the un-Higgsed branch a matching seems impossible due to the massless flavors.

Introducing the strong gauge dynamics, we fix the full effective superpotential to

$$W_{eff} = S \log \Lambda^{3N_c-1} - (N_c - 1) S \log S + (N_c - 1) S - \\ - \frac{\lambda^2}{8\alpha} \mp \frac{\lambda^2}{8\alpha} \sqrt{1 + \frac{4\alpha}{\lambda^2} S} + \frac{1}{2} S \log \alpha - \frac{1}{2} S \log \lambda - \\ - \frac{1}{2} S \log \left( \mp \sqrt{1 + \frac{4\alpha}{\lambda^2} S} - 1 \right) + S \log 2 + \frac{S}{4} , \quad (3.47)$$

where the upper sign corresponds to the Higgsed branch and the lower sign to the un-Higgsed one.

We want to recover the effective superpotential for the case  $\alpha \rightarrow 0$ . This limit is not sensible on the Higgsed branch since its vacua run off to infinity. The crucial ingredient is that the full superpotential also knows about the un-Higgsed branch, so we can just change the branch and take the limit  $\alpha \rightarrow 0$  there. If we do that we indeed recover the result (3.23). This approach will be used later for the chiral model and models with dynamical supersymmetry breaking.

## 4 Gauge Group $G_2$ with Three Flavors

In this subsection we will study  $\mathcal{N} = 1$  SQCD with exceptional gauge group  $G_2$ . We will concentrate on the case with three flavors in the real fundamental **7** representation. This case is instructive because it requires the introduction of a baryon operator in addition to mesons, it has an instanton generated superpotential [34, 35] and exhibits an interesting vacuum structure [36].

#### 4.1 The Effective Superpotential

Using the primitive invariants of  $G_2$  we can construct seven gauge invariant operators. Six of them correspond to mesons

$$X_{IJ} = \delta^{ij} Q_I^i Q_J^j , \quad (4.1)$$

where  $X_{IJ}$  is a symmetric matrix, and the seventh operator is the baryon

$$Z = \psi^{ijk} \epsilon_{IJK} Q_I^i Q_J^j Q_K^k , \quad (4.2)$$

where  $\psi^{ijk}$  is the  $G_2$  invariant three-tensor which also appears in the multiplication table of imaginary octonions. Note that capital letters,  $I, J, K = 1, 2, 3$ , denote flavor indices whereas small letters,  $i, j, k = 1, \dots, 7$ , denote gauge indices.

For vanishing tree level superpotential the classical theory possesses a  $U(3)$  flavor symmetry. For concreteness we will study the theory in the presence of the tree level superpotential

$$W_{tree} = m^{IJ} X_{IJ} + \lambda Z . \quad (4.3)$$

By a flavor rotation we can always make the mass matrix  $m^{IJ}$  diagonal, but as a further simplification we assume that all masses are equal

$$m_{IJ} = m \delta_{IJ} \quad (4.4)$$

which leaves a  $SO(3)$  flavor symmetry unbroken. From here on we will, therefore, consider the  $I, J$  indices as  $SO(3)$  indices.

Let us now analyze the extrema of the model with this tree-level superpotential at the classical level. The F-term constraints read

$$Q_I^i = -\frac{3\lambda}{2m} \psi^{ijk} \epsilon_{IJK} Q_J^j Q_K^k , \quad (4.5)$$

whereas the D-term constraints are

$$\delta_{IJ} \bar{Q}_I^i T_{ij}^a Q_J^j = 0 . \quad (4.6)$$

The  $T^a$  are generators of  $G_2$  in the fundamental representation. They furnish a subset of the  $SO(7)$  generators and hence are anti-symmetric in  $i, j$ . There are two solutions to (4.5) and (4.6): in the first  $Q_I^i = 0$  and the gauge symmetry is unbroken, in the second  $Q_I^i = -\frac{2m}{3\lambda} \delta_I^i$ , after flavor and gauge transformations, which leaves an  $SU(2)$  gauge symmetry unbroken. In the semi-classical regime the matter fields are all heavy and can be integrated so that the total number of quantum vacua is the sum of the Witten indices of  $G_2$  and  $SU(2)$  SYM. This means that we expect six confining vacua with broken chiral symmetry.

In order to construct the Konishi anomaly relations and its generalizations we have to consider two kinds of transformations:

$$Q_I^i \rightarrow Q_I^i + \epsilon \lambda_I^j Q_J^i, \quad (4.7)$$

and

$$Q_I^i \rightarrow Q_I^i + \epsilon \epsilon_{IJK} \psi^{ijk} Q_J^j Q_K^k. \quad (4.8)$$

By contraction of the variations with  $\partial W / \partial Q_I^i$  we obtain the tree-level contributions to the Konishi anomaly which can be expressed as classical constraints for the gauge invariant meson and baryon operators

$$2mX^{IJ} + \lambda \delta^{IJ} Z = 0, \quad (4.9)$$

and

$$6\lambda((X_I^I)^2 - X^{IJ}X_{IJ}) + 2mZ = 0. \quad (4.10)$$

In these variables the two vacuum solutions turn out to be

$$X^{IJ} = Z = 0, \quad (4.11)$$

and

$$X^{II} = m^2/\lambda^2, \quad X^{IJ} = 0 \text{ for } I \neq J, \quad Z = -2m^3/\lambda^3. \quad (4.12)$$

Taking into account the one-loop exact correction to the Konishi anomaly we find<sup>5</sup>

$$\begin{aligned} 2mX^{IJ} + \lambda \delta^{IJ} Z &= 2S, \\ 6\lambda((X_I^I)^2 - X^{IJ}X_{IJ}) + 2mZ &= 0. \end{aligned} \quad (4.13)$$

In particular the second line, which corresponds to a generalized Konishi anomaly, does not receive quantum corrections in this particular case. The two solution to these quantum relations are:

$$X^{IJ} = x\delta^{IJ}, \quad Z = -18\frac{\lambda}{m}x^2, \quad x_{\pm} = -\frac{1}{18\lambda^2} \left( -m^2 \pm \sqrt{m^4 - 36\lambda^2 m S} \right), \quad (4.14)$$

where the  $x = x_+$  corresponds to the classical vacuum with vanishing vevs and  $x = x_-$  corresponds to the Higgsed vacuum with non-zero vevs. Using

$$\frac{\partial W_{eff}}{\partial m} = 3\langle x \rangle_S \quad \text{and} \quad \frac{\partial W_{eff}}{\partial \lambda} = \langle Z \rangle_S = -18\frac{\lambda}{m}\langle x \rangle_S^2, \quad (4.15)$$

we can solve for the perturbative part of the effective superpotential

$$\begin{aligned} W_{eff} &= \frac{m^3}{18\lambda^2} \left( 1 \mp \sqrt{1 - \frac{36\lambda^2}{m^3} S} \right) + \\ &\quad 2S \log \left( 1 \pm \sqrt{1 - \frac{36\lambda^2}{m^3} S} \right) + 3S \log m^3 + C(S). \end{aligned} \quad (4.16)$$

---

<sup>5</sup>Note that the 2 in front of the gluino operator  $S$  is due to the fact that the index of the fundamental 7 representation of  $G_2$  is 2, whereas the fundamental of  $SU(N)$  has index 1.

Matching this in the UV to the  $G_2$  theory with three fundamental chiral multiplets we can fix  $C(S)$ . We find

$$W_{eff} = 4S \left( -\log \frac{S}{\hat{\Lambda}^3} + 1 \right) + \frac{m^3}{18\lambda^2} \left( 1 - \sqrt{1 - \frac{36\lambda^2}{m^3} S} \right) - S + 2S \log \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{36\lambda^2}{m^3} S} \right), \quad (4.17)$$

where  $\hat{\Lambda}^{12} = \Lambda^9 m^3$ . Looking at the leading  $S \log S$  terms for the two possible branches of the square roots in (4.17) we find 4 + 2 extrema, hence, there are six vacua as expected [36].

## 4.2 The Vacuum Structure

The quantum vacuum manifold is described by the extrema of (4.17). Combining both branches one is led to the polynomial equation

$$\left( \hat{S}^3 + \hat{S} + \frac{z}{2} \right) \left( \hat{S}^3 - \hat{S} + \frac{z}{2} \right) = 0, \quad \hat{S} = \frac{S}{\hat{\Lambda}^3}, \quad z = \hat{\Lambda}^3 \frac{18\lambda^2}{m^3}. \quad (4.18)$$

The complex surface defined by equation (4.18) has six sheets and there are branch points at  $z = \infty$  and at the roots of  $z^4 = 256/729$ .

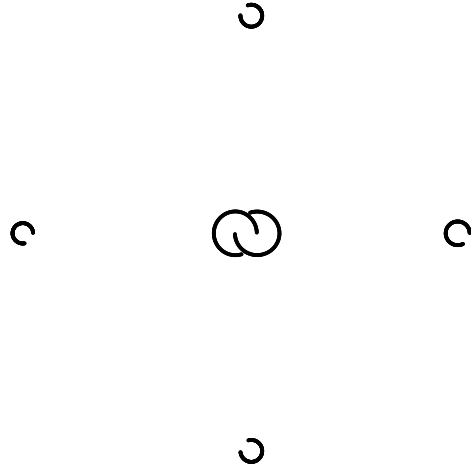


Figure 3: The trivial monodromy around  $z = 0$  as seen in the  $\hat{S}$  plane. The un-Higgsed vacua, corresponding to the smaller outer circles, are not interchanged, whereas the two circles in the middle correspond to the two Higgsed vacua which move around each other but come back to itself.

In order to understand the physics near the critical points we study the monodromies around them. The situation near  $z = 0$  is depicted in Fig. 3.

More precisely we are taking a small loop around  $z = 0$  and see, contrary to the case of SQCD, that the monodromy is trivial. The four un-Higgsed vacua of the unbroken  $G_2$  theory correspond to the four outer vacua and stay where they are. The two vacua in the middle of the picture, which correspond to the vacua where  $G_2$  is Higgsed to  $SU(2)$ <sup>6</sup>, loop around each other but eventually return to their starting positions. In the limit  $z \rightarrow 0$  the gluino condensate  $S$  goes to zero, but this is not related to the appearance of a chirally symmetric vacuum. Actually, the two vacua show run-away behavior as can be seen by inspecting the meson vevs and  $W_{eff}$  which both are driven to infinity in this limit.

More interesting are the critical points located at the roots of  $z^4 = 256/729$ . In Fig. 4 it can be seen that three of the un-Higgsed  $G_2$  vacua (outer circles) and one of the Higgsed vacua (circle in the center) remain at their original location. However, in the big circle on the right hand side of the picture we see the exchange of one Higgsed with one un-Higgsed vacuum.

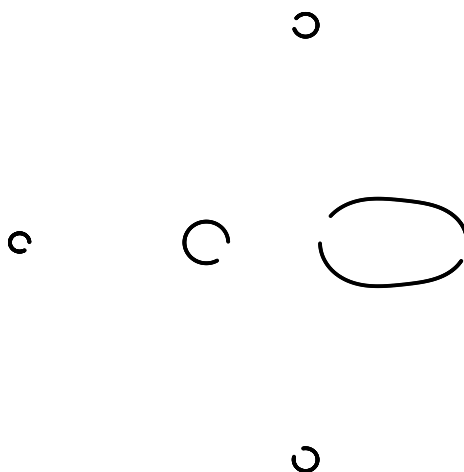


Figure 4: Monodromy around  $z = 4/\sqrt{27}$  as seen in the  $\hat{S}$  plane. The un-Higgsed vacua correspond to the smaller outer circles, whereas the bigger inner circle corresponds to a Higgsed vacuum. Most interesting is the rightmost big circle in which we can see the exchange of a un-Higgsed with a Higgsed vacuum.

Finally, we discuss the monodromy at  $z = \infty$ . For large  $z$  the six confining, chiral symmetry breaking vacua arrange themselves symmetrically on a circle. However, the monodromy  $z \rightarrow ze^{2\pi i}$  does not exchange the vacua in a  $Z_6$  symmetric fashion but rather acts like a  $Z_3$  rotation on two groups of three vacua, *i.e.* vacua are simultaneously exchanged in the sequence  $(1 \rightarrow 3 \rightarrow 5 \rightarrow 1)$  and  $(2 \rightarrow 4 \rightarrow 6 \rightarrow 2)$ . This behavior can also be anticipated by the fact that the equation for the quantum parameter space (4.18) factorizes. To summarize, the combined actions of the monodromies

<sup>6</sup>Generically this  $SU(2)$  theory confines and has two vacua, but we call them Higgsed vacua here to distinguish them.

permute the six vacua which are organized in two groups of three but does not lead to exchanges between the two groups.

## 5 A Chiral $SU(6)$ Model

In this section we apply the developed methods to chiral theories. We will start with a model that has well-defined supersymmetric ground states at the quantum level, and defer the study of the interesting case of dynamical supersymmetry breaking to a later section.

For concreteness we consider the case of  $SU(6)$  with two antifundamentals  $\bar{Q}^I$  and one antisymmetric tensor  $X$  [37, 38, 39, 40]. The relevant gauge invariant operators are

$$T = \varepsilon_{IJ} \bar{Q}_i^I \bar{Q}_j^J X^{ij} , \quad (5.1)$$

and

$$U = PfX = X^{i_1 j_1} X^{i_2 j_2} X^{i_3 j_3} \varepsilon_{i_1 j_1 i_2 j_2 i_3 j_3} , \quad (5.2)$$

where the capital  $I, J = 1, 2$  denote flavor indices and the small  $i, j$  denote color indices. We want to find the effective superpotential for a theory with the following tree level superpotential

$$W_{tree} = hT + gU . \quad (5.3)$$

The classical vacuum is given by  $T = U = 0$ . Since this is a chiral model we cannot introduce mass terms for the matter fields. We thus cannot separate the gauge dynamics from the dynamics of the light fields at a perturbative level and the matching to a one loop calculation will in general fail. Therefore, we try the technique we have already applied successfully to the massless SQCD case in section 3.4. A deformation of the tree level superpotential will give a classical vacuum where the gauge group is Higgsed. If all the light matter degrees of freedom are eaten up by the Higgs mechanism we can reliably match the effective superpotential below the Higgsing scale to a one loop calculation in the remaining gauge group.

We will argue that the following deformation of the tree level superpotential will give us a classical vacuum with the desired properties

$$W_{tree} = hT + gU + \lambda TU . \quad (5.4)$$

The classical vacua have to satisfy the F-flatness conditions

$$\begin{aligned} hT + \lambda TU &= 0 , \\ hT + 3gU + 4\lambda TU &= 0 , \end{aligned} \quad (5.5)$$

which have two solutions  $T = U = 0$  and  $T = -\frac{g}{\lambda}$ ,  $U = -\frac{h}{\lambda}$ . In addition we have to satisfy the D-flatness conditions

$$\bar{Q}_I^\dagger \bar{Q}_i^I - \frac{\delta_i^j}{6} \bar{Q}_K^\dagger \bar{Q}_k^K = 2X^\dagger_{ik} X^{kj} - \frac{\delta_i^j}{3} X^\dagger_{kl} X^{lk} , \quad (5.6)$$

where we have taken into account that the  $SU(6)$  generators are traceless. Using gauge and  $SU(2)$  flavor rotations we can parametrize the solution to these equations as follows:

$$\bar{Q} = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & -\lambda_2 & 0 \end{pmatrix}, \quad (5.7)$$

where D-flatness requires  $|\rho|^2 = 2|\lambda_1|^2 - 2|\lambda_2|^2$ .

From this we can see easily that the gauge group is broken to  $Sp(2)$ . First the  $\bar{Q}$  break  $SU(6) \rightarrow SU(4)$  and the subgroup of  $SU(4)$  that leaves invariant the lower four-by-four block of  $X$  is  $Sp(2)$ . At the same time 25 out of the 27 matter fields are eaten up by the Higgs mechanism. The two remaining matter fields are massive singlets under the  $Sp(2)$ . Hence, at low energies the theory becomes pure glue.

The anomalous Konishi variations give

$$\begin{aligned} hT + \lambda TU &= S, \\ hT + 3gU + 4\lambda TU &= 4S, \end{aligned} \quad (5.8)$$

with  $S = -\frac{1}{32\pi^2} \text{tr}_f W_\alpha W^\alpha$  such that we have

$$\begin{aligned} T &= \frac{g}{2\lambda} \left( -1 \pm \sqrt{1 + \frac{4\lambda}{gh} S} \right), \\ U &= \frac{h}{2\lambda} \left( -1 \pm \sqrt{1 + \frac{4\lambda}{gh} S} \right). \end{aligned} \quad (5.9)$$

The perturbative part of the effective superpotential can then again be determined by using gradient equations, it turns out to be

$$W_{eff} = \frac{gh}{2\lambda} \left( -1 \pm \sqrt{1 + \frac{4\lambda}{gh} S} \right) + S \log gh + 2S \log \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{4\lambda}{gh} S} \right) + C(S). \quad (5.10)$$

As can be seen from (5.9) the upper (+) sign corresponds to the un-Higgsed branch, whereas the lower (−) sign corresponds to the Higgsed branch.

To determine  $C(S)$  we match the effective superpotential to the Veneziano-Yankielowicz potential of the (classically) unbroken gauge group on the Higgsed branch. We know that on the Higgsed branch below the Higgsing scale the theory is described by pure  $Sp(2)$  glue. If we take for the purpose



of matching the bare parameters  $g = h$  then the unique Higgsing scale is  $\Delta = T^{1/3} = U^{1/3} = (-g/\lambda)^{1/3}$ . The strong coupling dynamics of  $Sp(2)$  are described by

$$W_{VY} = 3S \left( -\log \frac{S}{\Lambda_2^3} + 1 \right) , \quad (5.11)$$

where the scale  $\Lambda_2$  of the  $Sp(2)$  is related to the scale  $\Lambda_6$  of the  $SU(6)$  by the usual matching of scales at the Higgsing scale  $\Delta$  as

$$\left( \frac{\Lambda_6}{\Delta} \right)^{15} = \left( \frac{\Lambda_2}{\Delta} \right)^9 . \quad (5.12)$$

Demanding that the effective superpotential reproduce the potential (5.11) below  $O(S^2)$  we fix the  $C(S)$ . We find the full effective superpotential

$$\begin{aligned} W_{eff} = & -5S \log \frac{S}{\Lambda_6^3} + 4S - 2S \log 2 + \\ & + S \log gh + \frac{gh}{2\lambda} \left( -1 \pm \sqrt{1 + \frac{4\lambda}{gh} S} \right) + \\ & + 2S \log \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + \frac{4\lambda}{gh} S} \right) . \end{aligned} \quad (5.13)$$

If we want to recover the superpotential for the theory with  $\lambda = 0$  we have to go to the un-Higgsed branch which corresponds to the upper (+) sign. We emphasize again that we have used the semiclassical region of the Higgsed branch to determine the integration “constant”  $C(S)$ , but once we have obtained the full effective superpotential we can use its analytic structure and move freely among the branches.

In the  $\lambda \rightarrow 0$  limit on the un-Higgsed branch we thus obtain

$$W_{eff} = 5S \left( -\log \frac{S}{\Lambda_6^3} + 1 \right) + S \log gh . \quad (5.14)$$

This effective superpotential precisely reproduces the 5 vacua which were found e.g. in [40] by instanton calculations

$$S = \Lambda_6^3 (gh)^{\frac{1}{5}} e^{2\pi i k/5} . \quad (5.15)$$

## 6 Dynamical SUSY Breaking

We now want to see how the chiral ring and the Konishi anomaly can be used to understand theories with dynamical supersymmetry breaking. Naively, we cannot apply the method we used in the previous sections, since there

is no supersymmetric vacuum. However, one can deform the original tree level superpotential to get supersymmetric vacua in which we can make a reliable calculation of the effective superpotential. One can then analyze the behavior of the vacua when switching the deformation off again.

To be specific we discuss a variant of the Izawa-Yanagida-Intriligator-Thomas (IYIT) model [41, 42] which features dynamical supersymmetry breaking (DSB). The model is  $\mathcal{N} = 1$  supersymmetric  $Sp(N_c)$  gauge theory with  $2N_f = 2(N_c + 1)$  fundamental chiral multiplets  $Q_a^i$ , ( $a = 1, \dots, 2N_c$ ,  $i = 1, \dots, 2N_f$ ), and a gauge singlet chiral multiplet  $S_{ij}$ , which is antisymmetric in the indices  $i, j$ . The gauge invariant matter fields of this theory are the meson  $M^{ij} = Q^i Q^j$  and  $S_{ij}$ . Note that we will denote both the gauge singlet and the glueball superfield by  $S$ . However, the former always carries flavor indices, so no confusion should arise. We consider the above theory with a tree level potential given by

$$W_{\text{tree}} = \lambda S_{ij} M^{ij} - m J^{ij} S_{ij} , \quad (6.1)$$

where  $J = \mathbb{1}_{N_f} \otimes i\sigma^2$  is the symplectic form. This model has been studied in [43]. The effective superpotential is

$$W_{\text{eff}} = X (Pf M - \Lambda^{2N_c+2}) + \lambda S_{ij} M^{ij} - m J^{ij} S_{ij} . \quad (6.2)$$

Integrating out the Lagrange multiplier field  $X$  and the gauge singlet  $S_{ij}$  we get<sup>7</sup>

$$Pf M = \Lambda^{2N_c+2}, \quad M = \frac{m}{\lambda} J . \quad (6.3)$$

From these equations it follows that the above superpotential has a minimum only when

$$\left(\frac{m}{\lambda}\right)^{N_c+1} = \Lambda^{2N_c+2} . \quad (6.4)$$

This is the condition on the bare parameters for unbroken supersymmetry.

## 6.1 Massive Deformation

We now turn to the problem to derive (6.4) without knowing the superpotential (6.2). In order to guarantee the existence of a supersymmetric vacuum, we can deform the tree level superpotential in such a way that there is a classical vacuum in which all matter becomes massive. This allows to implement the strong IR dynamics and the full quantum theory has supersymmetric vacua. When turning those deformations off, we will see, that all the vacua run away, unless (6.4) is satisfied.

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<sup>7</sup>The Pfaffian of  $M$  is defined here as  $Pf M = \frac{1}{2^N N_f!} \epsilon_{i_1 j_1, \dots, j_{N_f}} M^{i_1 j_1} \dots M^{i_{N_f} j_{N_f}}$ , such that  $Pf J = 1$ .

We deform the above theory by giving the gauge singlets  $S_{ij}$  a mass  $\alpha S_{ij} S^{ij}$ . The tree level superpotential is

$$W_{tree} = \lambda S_{ij} M^{ij} - m J^{ij} S_{ij} + \alpha S_{ij} S^{ij} , \quad (6.5)$$

where  $S^{kl} = S_{ij} J^{ik} J^{jl}$ . This potential has many vacua. Two of them will be important in the following. The vacuum with all squarks massless at  $M^{ij} = \frac{m}{\lambda} J^{ij}$  and  $S_{ij} = 0$ . It exists also for  $\alpha = 0$ . The other one is the massive vacuum, with  $Q = 0$  and  $S_{ij} = -\frac{m}{2\alpha} (J^{-1})_{ij}$ . Around the second classical vacuum we can integrate out the massive fields, such that in the IR we are left with pure gauge theory.

To obtain the Konishi relations we consider the variations  $\delta_1 Q^i = \epsilon_j^i Q^j$ , and  $\delta_2 S_{ij} = \epsilon_{ij}^{lm} S_{lm}$ . These give rise to the following respective relations

$$\begin{aligned} 2\lambda \langle S_{ij} M^{kj} \rangle_S &= \delta_i^k S , \\ \lambda \langle S_{ij} M^{kl} \rangle_S - m J^{kl} \langle S_{ij} \rangle_S + 2\alpha \langle S^{kl} \rangle_S \langle S_{ij} \rangle_S &= 0 . \end{aligned} \quad (6.6)$$

After straightforward algebra one can solve for the expectation values of the chiral operators,

$$\begin{aligned} S_{ij} &= (A \otimes i\sigma^2)_{ij} , \quad A_{ab} = \delta_{ab} \left( \frac{m}{4\alpha} - \eta_a \sqrt{\frac{m^2}{16\alpha^2} - \frac{S}{4\alpha}} \right) , \quad a, b = 1, \dots, N_f , \\ M^{ij} &= (m J^{ij} - 2\alpha S^{ij}) / \lambda , \end{aligned} \quad (6.7)$$

where we used flavor rotations to bring  $A$  to diagonal form and  $\eta_a$  denotes a choice of a vector with components  $\pm 1$ . The number of  $\pm 1$  entries in  $\eta_a$  will be denoted by  $N_f^\pm$ . Note that  $N_f^+ = 0$  corresponds to the massive vacuum, as can be seen in the classical limit of  $S$  small.  $N_f^+ = N_f$  corresponds to the classical vacuum, which exists also for  $\alpha = 0$ .

The expectation values of the chiral composites then read,

$$\begin{aligned} \langle S_{ij} M^{ij} \rangle_S &= N_f \frac{S}{\lambda} , \\ \langle J^{ij} S_{ij} \rangle_S &= N_f^+ \frac{m}{\alpha} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\alpha S}{m^2}} \right) + N_f^- \frac{m}{\alpha} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\alpha S}{m^2}} \right) , \\ \langle S_{ij} S^{ij} \rangle_S &= N_f^+ \frac{m^2}{2\alpha^2} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\alpha S}{m^2}} \right)^2 + N_f^- \frac{m^2}{2\alpha^2} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\alpha S}{m^2}} \right)^2 . \end{aligned} \quad (6.8)$$

Integrations with respect to the various parameters then gives the perturbative part of the effective superpotential  $W_{eff}^{pert}$ . By matching  $W_{eff}^{pert}$  to the VY potential, which describes the pure gauge dynamics around the classical

vacuum  $Q = 0$  and  $S_{ij} = -\frac{m}{2\alpha}(J^{-1})_{ij}$ , gives the following effective superpotential,

$$\begin{aligned} W_{eff} = & (N_f) \left[ S \log \frac{\Lambda^3}{S} + 1 \right] + N_f S \log \left( \frac{\lambda S}{\Lambda m} \right) - N_f \left( \frac{m^2}{4\alpha} + \frac{S}{2} \right) \\ & + (N_f^+ - N_f^-) \frac{m^2}{4\alpha} \sqrt{1 - \frac{4\alpha}{m^2} S} - \\ & - S \log \left[ \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\alpha}{m^2} S} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\alpha}{m^2} S} \right)^{N_f^-} \right]. \end{aligned} \quad (6.9)$$

The derivative  $\partial_S W_{eff}(S) = 0$  then leads to

$$\log \left[ \left( \frac{m}{\Lambda^2 \lambda} \right)^{N_f} r \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4\alpha}{m^2} S} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4\alpha}{m^2} S} \right)^{N_f^-} \right] = 0. \quad (6.10)$$

For  $N_f^- = N_f$  this can be solved explicitly to give

$$\begin{aligned} S &= e^{2\pi i k / (N_c + 1)} \Lambda^2 \frac{m \lambda}{\alpha} \left( 1 - e^{2\pi i k / (N_c + 1)} \frac{\Lambda^2 \lambda}{m} \right), \\ S_{ij} &= -\frac{m}{2\alpha} \left( 1 - e^{2\pi i k / (N_c + 1)} \frac{\Lambda^2 \lambda}{m} \right) (J^{-1})_{ji}, \\ M^{ij} &= e^{2\pi i k / (N_c + 1)} \Lambda^2 J^{ji}. \end{aligned} \quad (6.11)$$

In total we find  $N_c + 1$  vacua for each point of the parameter space. The massive vacuum can be found in the limit of small  $\Lambda^2 \lambda / m$ . The classical vacuum which exists also for  $\alpha = 0$  corresponds to the points  $m / \lambda \Lambda^2 = e^{2\pi i k / (N_c + 1)}$ .

For generic points in the parameter space with  $N_f^-$  arbitrary the zero mass limit gives run-away vacua. Only for  $N_f^- = N_c + 1$  and  $m / \lambda \Lambda^2 = e^{2\pi i k / (N_c + 1)}$  we find finite expectation values in the  $\alpha \rightarrow 0$  limit. The vacuum with  $S = S_{ij} = 0$  and  $M^{ij} = e^{2\pi i k / (N_c + 1)} \Lambda^2 J^{ij}$  stays finite, whereas the  $N_c$  other vacua still run away. We recover the quantum constraint on the parameters (6.4).

As is by now familiar we expect to recover (6.4) directly if we calculate the effective superpotential for the theory with  $\alpha = 0$  by switching branches and taking the limit  $\alpha \rightarrow 0$ . The other branch corresponds to  $N_f^+ = N_f$  and sending  $\alpha$  to zero gives

$$W_{eff} = N_f S \log \frac{\Lambda^2 \lambda}{m}. \quad (6.12)$$

It is interesting that  $S$  appears just as a Lagrange multiplier for the constraint (6.4) in this superpotential. A similar appearance of  $S$  in the effective superpotential was observed in [18].

This method can be generalized to other models with dynamical supersymmetry breaking. A mass term will typically produce a supersymmetric quantum vacuum. In the limit of turning off the mass, one can see how the quantum vacua run away, except for some vacua, which might stay finite for certain choices of the other parameters. We can see here the mechanism by which the Witten index jumps, when the highest couplings in the tree level superpotential are switched off.

## 6.2 An Alternative Derivation

In this section we will derive (6.4) using the Konishi anomaly without using a mass deformation. Our strategy will be to assume unbroken supersymmetry, so that we can use the Konishi anomaly relations and derive an effective superpotential as a function of  $S$ . Minimizing this effective superpotential with respect to  $S$  should lead to (6.4).

To obtain the Konishi relations we consider the variations  $\delta_1 Q^i = \epsilon_j^i Q^j$ , and  $\delta_2 S_{ij} = \epsilon_{ij}{}^{lm} S_{lm}$ . These give rise to the following respective relations

$$\begin{aligned} 2\lambda \langle S_{ij} M^{kj} \rangle_S &= \delta_i^k S, \\ \lambda \langle S_{ij} M^{kl} \rangle_S - m J^{kl} \langle S_{ij} \rangle_S &= 0. \end{aligned} \quad (6.13)$$

The equations (6.13) contain a lot of valuable information. As usual, they enable us to derive the dependence of  $W_{eff}$  on the bare parameters. Rewriting (6.13) as

$$\begin{aligned} \frac{\partial W_{eff}}{\partial \lambda} &= N_f \frac{S}{\lambda}, \\ \lambda \frac{\partial W_{eff}}{\partial \lambda} + m \frac{\partial W_{eff}}{\partial m} &= 0, \end{aligned} \quad (6.14)$$

we can solve for  $W_{eff}$  to get

$$W_{eff}(S, \lambda, m) = N_f S \log \frac{\lambda}{m} + C(S). \quad (6.15)$$

However, due to the factorization of the chiral vevs we can also rewrite the conditions in (6.13) as

$$\begin{aligned} \lambda \langle S_{ij} \rangle_S \langle M^{kj} \rangle_S &= \delta_i^k S, \\ \lambda \langle S_{ij} \rangle_S \langle M^{kl} \rangle_S - m J^{kl} \langle S_{ij} \rangle_S &= 0, \end{aligned} \quad (6.16)$$

and solve for  $\langle S_{ij} \rangle_S$  and  $\langle M^{ij} \rangle_S$ . We get

$$\langle M^{ij} \rangle_S = \frac{m}{\lambda} J^{ij}, \quad \langle S_{ij} \rangle_S = \frac{1}{2m} (J^{-1})_{ji}. \quad (6.17)$$

We have now gathered enough information to turn to the derivation of the full  $W_{eff}(S, \lambda, m)$ . First, we think of  $\lambda S_{ij}$  as a mass for the fundamental chiral multiplets and integrate them out. This will give us an effective superpotential as a function of  $(S, S_{ij}, \lambda, m)$ . Note that in this model the canonical mass term for the fundamentals is  $\frac{1}{2}m_{ij}Q^iQ^j$  such that the canonical mass is expressed as  $m_{ij} = 2\lambda S_{ij}$ . A perturbative evaluation then yields

$$W_{eff}^{pert} = 3(N_c + 1)S \log \frac{\Lambda}{\mu} + S \log Pf \left( \frac{2\lambda}{\Lambda} S_{ij} \right) + W^{(1)}(S, S_{ij}, \lambda, m) . \quad (6.18)$$

Note that we have already taken into account the contribution of the bare coupling  $\tau$  enabling us to replace the UV cutoff by the dynamically generated scale. The part  $W^{(1)}$  will be determined by the requirements that (a) the extremal value of  $S_{ij}$  satisfy the second equation of (6.17) and that (b) the superpotential obtained after integrating out  $S_{ij}$  have the appropriate dependence on the bare parameters (6.15). Both requirements uniquely fix the effective action to

$$W_{eff}^{pert} = 3(N_c + 1)S \log \frac{\Lambda}{\mu} + S \log Pf \left( \frac{2\lambda}{\Lambda} S_{ij} \right) - m J^{ij} S_{ij} . \quad (6.19)$$

Indeed, after integrating out  $S_{ij}$  and replacing the first part by the appropriate Veneziano-Yankielowicz term, we find

$$W_{eff} = (N_c + 1)S \left[ \log \frac{\Lambda^3}{S} + 1 \right] + S \log Pf \left[ \frac{\lambda S}{m\Lambda} (J^{-1})_{ji} \right] - N_f S , \quad (6.20)$$

which has the appropriate dependence on the bare parameters. After taking into account that  $N_f = N_c + 1$  the effective superpotential simplifies to

$$W_{eff} = (N_c + 1)S \log \frac{\lambda \Lambda^2}{m} . \quad (6.21)$$

This is the expected expression (6.12). Minimizing with respect to  $S$  gives the relation

$$\left( \frac{m}{\lambda} \right)^{N_c+1} = \Lambda^{2(N_c+1)} . \quad (6.22)$$

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## A Proof of One Loop Exactness of the Konishi Anomaly

In this appendix we want to show how the proof for the one loop exactness of the generalized Konishi anomaly works. We follow the idea of [5]. We concentrate on the normal Konishi anomaly for SQCD with one flavor for concreteness. It is easy to generalize this proof to other cases.

There are two flavor symmetries  $U(1)_Q$  and  $U(1)_{\tilde{Q}}$  together with an R-symmetry  $U(1)_R$ . Those symmetries are broken by the tree level superpotential and by anomalies. By promoting the coupling constants to chiral superfields, which transform under those symmetries, we can restore those symmetries. The charges are summarized in the following table

	$U(1)_Q$	$U(1)_{\tilde{Q}}$	$U(1)_R$	
$Q$	1	0	$\frac{2}{3}$	
$\tilde{Q}$	0	1	$\frac{2}{3}$	
$W_\alpha$	0	0	1	
$m$	-1	-1	$\frac{2}{3}$	
$\lambda$	-2	-2	$-\frac{2}{3}$	
$\Lambda^{3N-1}$	1	1	$2N - \frac{2}{3}$	(A.1)

We want to calculate the lowest component of the Konishi anomalies for

$$Q \mapsto Q + \epsilon Q \quad \text{and} \quad \tilde{Q} \mapsto \tilde{Q} + \tilde{\epsilon} \tilde{Q}. \quad (\text{A.2})$$

Let us concentrate on the first Konishi anomaly. We want to calculate the divergence of the supercurrent associated with the first transformation in (A.2)

$$\bar{D}^2 J = \bar{D}^2 Q^\dagger e^V Q = \frac{\partial W_{tree}}{\partial q} q + \mathcal{O}(\theta, \bar{\theta}) \quad (\text{A.3})$$

in a slowly varying background gaugino field. The lowest component of this expression is a chiral operator. This chiral operator depends (modulo  $\bar{Q}$  exact operators) only on other chiral operators and it depends only holomorphically on the coupling constants. Furthermore we assume, a smooth weak coupling behavior, *i.e.* the coupling constants can only appear with positive integer powers. The scale  $\Lambda$  can only appear in positive integer powers of  $\Lambda^{3N-1}$ , because the leading non-perturbative effects at weak coupling are due regular instantons and, in particular, we do not expect fractional instantons to contribute. We also assume, that all fields can only appear with positive powers, *i.e.* that there is no singularity at the origin in field space.

The divergence  $\bar{D}^2 J$  has the charges  $(0, 0, 2)$ . Charge conservation then gives constraints on the powers in which all the fields and coupling constants

can appear

$$n_Q \begin{pmatrix} 1 \\ 0 \\ \frac{2}{3} \end{pmatrix} + n_{\tilde{Q}} \begin{pmatrix} 0 \\ 1 \\ \frac{2}{3} \end{pmatrix} + n_{W_\alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + n_m \begin{pmatrix} -1 \\ -1 \\ \frac{2}{3} \end{pmatrix} + n_\lambda \begin{pmatrix} -2 \\ -2 \\ -\frac{2}{3} \end{pmatrix} + n_{\Lambda^{3N-1}} \begin{pmatrix} 1 \\ 1 \\ \frac{6N-2}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \quad (\text{A.4})$$

The first two equations imply  $n_{\tilde{Q}} = n_Q$ . Subtracting four times the first equation from the third equation, we get

$$n_{W_\alpha} + 2n_m + 2n_\lambda + (2N - 2)n_{\Lambda^{3N-1}} = 2. \quad (\text{A.5})$$

For  $N > 2$  this implies, that  $n_{\Lambda^{3N-1}} = 0$ , *i.e.* there are no nonperturbative contributions. This, together with the first equation of the charge conservation, leaves us with three kinds of solutions

	$n_Q$	$n_{\tilde{Q}}$	$n_{W_\alpha}$	$n_m$	$n_\lambda$	$n_{\Lambda^{3N-1}}$
1.	0	0	2	0	0	0
2.	1	1	0	1	0	0
3.	2	2	0	0	1	0

(A.6)

We now need to determine, which kinds of diagrams can contribute in each of those three cases. To this end we need to look at the Feynman rules. Since we want to calculate correlators with only chiral fields as external legs, which depend holomorphically on the coupling constants, there are only three kinds of vertices, that can contribute.

- The vertex of the current

$$\frac{\partial W_{\text{tree}}}{\partial \phi} \phi, \quad (\text{A.7})$$

- The vertex due to the superpotential

$$\frac{\partial^2 W_{\text{tree}}}{\partial \phi} \psi^2 \quad (\text{A.8})$$

- The coupling to the gaugino

$$\phi^\dagger W_\alpha \psi^\alpha. \quad (\text{A.9})$$

The first two kinds of vertices come with a coupling constant, whereas the third kind corresponds to the insertion of background gaugino field. Therefore, the number of vertices in a diagram is given by

$$V = \sum_j n_{g_j} + n_{W_\alpha}, \quad (\text{A.10})$$



where the coupling constants are denoted by  $g_j$ . The number of propagators can be determined by counting the number of internal legs on those vertices

$$P = \frac{1}{2} \left( \sum_j n_{g_j} l_j + 2n_{W_\alpha} - n_\phi \right), \quad (\text{A.11})$$

where  $l_j$  is the number of legs (bosonic and fermionic) on the vertex  $j$ . We can combine those two results to get the number of loops  $L$  in a diagram<sup>8</sup>

$$L = 1 + P - V = 1 + \frac{1}{2} \left( \sum_j n_{g_j} (l_j - 2) - n_\phi \right). \quad (\text{A.12})$$

Inserting the previous results (A.6) into this formula we see, that there are only tree level and one loop diagrams contributing to the lowest component of the Konishi anomaly, *i.e.* the anomaly is one loop exact and we can trust our expressions. This argument can easily be generalized to theories with different gauge groups and matter content, and also to the generalized Konishi anomaly. It is easy to see that a sufficient condition for the one loop exactness of the generalized Konishi anomaly is

$$2C(adj) - \sum_I 2C(r_I) > 2, \quad (\text{A.13})$$

where the sum is over all matter fields and  $2C(r)$  is the index of the representation  $r$ . This condition is satisfied in most of the cases we study. However, if (A.13) is not satisfied, one needs to study the full set of charge conservation equations, in analogy to (A.4). Sometimes the one loop exactness can fail, e.g. for too small gauge groups or for a sufficiently large number of external legs,  $\Lambda$  dependent terms can appear, which correspond to non-perturbative corrections to Konishi anomalies. We have not found an argument for the absence of such terms in general, but for the purpose of calculating superpotential such corrections do not appear in the examples studied in this paper.

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<sup>8</sup>This is the number of momentum loops, not the number of index loops in a ribbon graph.

## References

- [1] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” *Nucl. Phys.* **B644** (2002) 3–20, [hep-th/0206255](#).
- [2] R. Dijkgraaf and C. Vafa, “On geometry and matrix models,” *Nucl. Phys.* **B644** (2002) 21–39, [hep-th/0207106](#).
- [3] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” [hep-th/0208048](#).
- [4] G. Veneziano and S. Yankielowicz, “An effective lagrangian for the pure N=1 supersymmetric YANG-MILLS theory,” *Phys. Lett.* **B113** (1982) 231.
- [5] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, “Chiral rings and anomalies in supersymmetric gauge theory,” *JHEP* **12** (2002) 071, [hep-th/0211170](#).
- [6] E. Witten, “Chiral ring of  $\mathrm{Sp}(N)$  and  $\mathrm{SO}(N)$  supersymmetric gauge theory in four dimensions,” [hep-th/0302194](#).
- [7] K. Konishi, “Anomalous supersymmetry transformation of some composite operators in SQCD,” *Phys. Lett.* **B135** (1984) 439.
- [8] K.-i. Konishi and K.-i. Shizuya, “Functional integral approach to chiral anomalies in supersymmetric gauge theories,” *Nuovo Cim.* **A90** (1985) 111.
- [9] N. Seiberg, “Adding fundamental matter to ‘Chiral rings and anomalies in supersymmetric gauge theory’,” *JHEP* **01** (2003) 061, [hep-th/0212225](#).
- [10] R. Argurio, V. L. Campos, G. Ferretti, and R. Heise, “Exact superpotentials for theories with flavors via a matrix integral,” [hep-th/0210291](#).
- [11] J. McGreevy, “Adding flavor to Dijkgraaf-Vafa,” *JHEP* **01** (2003) 047, [hep-th/0211009](#).
- [12] H. Suzuki, “Perturbative derivation of exact superpotential for meson fields from matrix theories with one flavour,” [hep-th/0211052](#).
- [13] I. Bena and R. Roiban, “Exact superpotentials in  $N = 1$  theories with flavor and their matrix model formulation,” *Phys. Lett.* **B555** (2003) 117–125, [hep-th/0211075](#).
- [14] Y. Demasure and R. A. Janik, “Effective matter superpotentials from Wishart random matrices,” *Phys. Lett.* **B553** (2003) 105–108, [hep-th/0211082](#).

- [15] Y. Tachikawa, “Derivation of the konishi anomaly relation from Dijkgraaf-Vafa with (bi-)fundamental matters,” [hep-th/0211189](#).
- [16] R. Argurio, V. L. Campos, G. Ferretti, and R. Heise, “Baryonic corrections to superpotentials from perturbation theory,” *Phys. Lett. B* **553** (2003) 332–336, [hep-th/0211249](#).
- [17] S. G. Naculich, H. J. Schnitzer, and N. Wyllard, “Matrix model approach to the  $N = 2$   $U(N)$  gauge theory with matter in the fundamental representation,” *JHEP* **01** (2003) 015, [hep-th/0211254](#).
- [18] I. Bena, R. Roiban, and R. Tatar, “Baryons, boundaries and matrix models,” [hep-th/0211271](#).
- [19] B. Feng, “Seiberg duality in matrix model,” [hep-th/0211202](#).
- [20] B. Feng and Y.-H. He, “Seiberg duality in matrix models. II,” [hep-th/0211234](#).
- [21] Y. Ookouchi, “ $N = 1$  gauge theory with flavor from fluxes,” [hep-th/0211287](#).
- [22] K. Ohta, “Exact mesonic vacua from matrix models,” [hep-th/0212025](#).
- [23] I. Bena, S. de Haro, and R. Roiban, “Generalized Yukawa couplings and matrix models,” [hep-th/0212083](#).
- [24] C. Hofman, “Super Yang-Mills with flavors from large  $N(f)$  matrix models,” [hep-th/0212095](#).
- [25] R. Roiban, R. Tatar, and J. Walcher, “Massless flavor in geometry and matrix models,” [hep-th/0301217](#).
- [26] B. Feng, “Note on matrix model with massless flavors,” [hep-th/0212274](#).
- [27] Y. Demasure and R. A. Janik, “Explicit factorization of Seiberg-Witten curves with matter from random matrix models,” [hep-th/0212212](#).
- [28] H. Suzuki, “Mean-field approach to the derivation of baryon superpotential from matrix model,” [hep-th/0212121](#).
- [29] R. Dijkgraaf, M. T. Grisaru, C. S. Lam, C. Vafa and D. Zanon, “Perturbative computation of glueball superpotentials,” [arXiv:hep-th/0211017](#).
- [30] K. A. Intriligator, R. G. Leigh, and N. Seiberg, “Exact superpotentials in four-dimensions,” *Phys. Rev. D* **50** (1994) 1092–1104, [hep-th/9403198](#).
- [31] F. Ferrari, “Quantum parameter space and double scaling limits in  $N = 1$  super Yang-Mills theory,” [hep-th/0211069](#).

- [32] F. Cachazo, N. Seiberg, and E. Witten, “Phases of  $N = 1$  supersymmetric gauge theories and matrices,” **hep-th/0301006**.
- [33] F. Ferrari, “Quantum parameter space in super Yang-Mills. II,” **hep-th/0301157**.
- [34] I. Pesando, “Exact results for the supersymmetric  $G(2)$  gauge theories,” *Mod. Phys. Lett.* **A10** (1995) 1871–1886, **hep-th/9506139**.
- [35] S. B. Giddings and J. M. Pierre, “Some exact results in supersymmetric theories based on exceptional groups,” *Phys. Rev.* **D52** (1995) 6065–6073, **hep-th/9506196**.
- [36] A. V. Smilga, “ $6 + 1$  vacua in supersymmetric QCD with  $G(2)$  gauge group,” *Phys. Rev.* **D58** (1998) 105014, **hep-th/9801078**.
- [37] Y. Meurice and G. Veneziano, “SUSY vacua versus chiral fermions,” *Phys. Lett.* **B141** (1984) 69.
- [38] I. Affleck, M. Dine, and N. Seiberg, “Calculable nonperturbative supersymmetry breaking,” *Phys. Rev. Lett.* **52** (1984) 1677.
- [39] I. Affleck, M. Dine, and N. Seiberg, “Dynamical supersymmetry breaking in chiral theories,” *Phys. Lett.* **B137** (1984) 187.
- [40] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, “Nonperturbative aspects in supersymmetric gauge theories,” *Phys. Rept.* **162** (1988) 169–248.
- [41] K.-I. Izawa and T. Yanagida, “Dynamical Supersymmetry Breaking in Vector-like Gauge Theories,” *Prog. Theor. Phys.* **95** (1996) 829–830, **hep-th/9602180**.
- [42] K. A. Intriligator and S. Thomas, “Dynamical supersymmetry breaking on quantum moduli spaces,” *Nucl. Phys.* **B473** (1996) 121–142, **hep-th/9603158**.
- [43] J. de Boer, K. Hori, H. Ooguri, and Y. Oz, “Branes and dynamical supersymmetry breaking,” *Nucl. Phys.* **B522** (1998) 20–68, **hep-th/9801060**.